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Topological Foundations of the Marussi-Hotine Approach to Geodesy

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The usual formulation of the Hotine-Marussi approach to geodesy is essentially formal in character and makes no attempt to make precise the conditions under which the calculations are valid or address questions of existence and uniqueness. In the present paper we attempt to remedy this by translating various mathematical and physical requirements into topological assumptions. Topological methods are employed to study the family of

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equipotential surfaces Σ of the Earth's external gravity field. This leads to a clarification of the role of local coordinates on the surfaces of Σ, the imbedding of Σ in Euclidean 3-space, and the occurrence of singularities on surfaces of Σ.

CONTENTS

Preface

The following report is a revised version of an invited paper "The Mathematical Foundations of the Hotine-Marussi Approach to Geodesy" which was presented on June 5, 1989 at the Second Hotine-Marussi Symposium in Pisa. The talk in Pisa essentially consisted of an informal discussion of the basic ideas/results in the manuscript and was illustrated by examples and figures not included in the original paper. In the present report these examples and figures are included together with additional material derived from discussions with participants at the Symposium. This has resulted in almost a 50% increase in the size of the original manuscript.

The material addresses the topological foundations which underlie the entire Marussi-Hotine formulation of differential geodesy. It also furnishes an outline of some of the mathematical preliminaries required in the investigation of the hierarchy of questions related to the Marussi Hypothesis which is the primary topic of our research contract. A subsequent report, now in preparation, will be devoted to a mathematical appreciation of Marussi's contributions to geodesy and will contain an introduction to the Marussi Hypothesis in its original formulation.

1. Introduction

The Scottish mathematician and natural philosopher Sir James Ivory began one of his papers in *The Philosophical Magazine and Journal* for 1824 with the words:

> "It is not my intention to trace minutely the various labours of philosophers on the Figure of the Earth, but to state concisely the present mathematical theory on the subject, and to add some observations upon it."

Although I cannot aspire to attain either Ivory's eloquence, or lucidity of exposition, in the present report I will attempt to imitate his endeavors by an examination of mathematical foundations of the Marussi-Hotine approach to geodesy. This approach is based on an essential and extensive use of differential-geometric and tensor-theoretic methods which abandon the traditional views of H. Bruns (1878) and F. Helmert (1880) of geodesy as "the science of the measurement and mapping of the Earth's surface."

Indeed, in 1952 MARUSSI [1] summarized his approach by the statement that

> "Geodesy is the science which is devoted to the study of the Earth's gravity field."

From this viewpoint it follows that our primary object of concern is the *family* Σ of external equipotential surfaces of the Earth, and following F. Bocchio (1972) we will call the investigation of these surfaces *differential geodesy*. This terminology is slightly more general than Marussi's *intrinsic geodesy* [2], and less general than Hotine's *mathematical geodesy* [3]. Although Marussi's approach stressed the importance of conceptually formulating his theory in a *coordinate-free* manner, in practice much of his work was *coordinate-dependent*. He assumed the existence of an adequate supply

of natural physically relevant coordinates which he called *intrinsic coordinates*, and we term this assumption the *Marussi Hypothesis*. Hotine was less insistent and emphatic in this regard, but much of his analysis was inextricably tied to coordinates. However, it should be noted that some of his treatise foreshadows the *leg formalism* of E. Grafarend (1986), and B. Chovitz (1982) has pointed out that in his final work Hotine recognized the efficacy of differential forms. These were subsequently employed in geodesy by Grafarend (1971, 1975) and N. Grossman (1974) and their study of holonomic measurables has shed doubt on the general validity of the Marussi Hypothesis.

Despite its visionary nature the approach of Marussi and Hotine was essentially of a *formal* character, viz they did not make any attempt to seek either the precise conditions under which their calculations were valid or to address questions of existence and uniqueness. This is not intended to be a criticism, since virtually *all* pioneering work which applies mathematics to physical problems is formal. By necessity, pioneers are more concerned with the value and utility of an idea, and not with the ultimate consequences or restrictions under which their methods retain their meaning.

Our study is focussed on the members of the family Σ and what must be assumed — both mathematically and physically -- in order for Σ to be susceptible of geodetic and geometric study. In doing this we make no direct appeal to the physical equations whose solutions define Σ, and consequently our inquiry may be regarded as a geometric preliminary, or a *geometric constraint*, on the theory of the *geodetic boundary value problem*. The analytical aspects of this problem have been extensively investigated by Bjerhammar, Holota, Hörmander, Moritz, Sacerdote, Sansò, and Tscherning.

Our immediate concern is with the general properties of surfaces in Σ, and these include not only *local* properties where one deals with a piece of a equipotential surface S, but *global* properties characterizing the entire

surface S. We will see that both of these require topological notions. Indeed if differential geometry is regarded as providing a *geometric constraint* on questions in physical geodesy, then topology furnishes a *topological constraint* on differential geometry.

Topological considerations are perhaps new to mathematical geodesists; however, they have played a significant role in modern mathematics. They have allowed mathematicians to systematically dissect and analyze the fundamental notions of what is meant by the terms space, continuity, differentiability, etc. This has permitted one to recognize the aspects of these concepts which are strictly Euclidean as well as those which are more general. Indeed, in 1939 Hermann Weyl, one of the giants of twentieth century mathematics, remarked that

> "In these days the angel of topology and the devil of abstract algebra fight for the soul of each individual mathematical domain."

The last half-century of mathematical endeavors have vigorously confirmed Weyl's comments.

Mathematically such considerations are not new, and date back to Leibniz (1678), who proposed their investigation as *analysis situs*. This name was employed into the early part of the twentieth century, when the term *topology* came into popular use. It is amusing to note that the same man who coined the term geoid in 1873 invented the word topology in 1847. He also published an account of the *one-sided* Möbius strip four years before A.F. Möbius (1865)! This was Gauss' student J.B. Listing (1808-1882), and his "Vorstudien zur Topologie" was one of the earliest systematical treatments of the subject. Listing's view of topology as a

> "kalkulatorische Bearbeitung der modalen Seite der Geometrie,"

is in a sense how we will seek to apply it to differential geodesy.

In differential geodesy *two* important types of assumptions arise when one

attempts to pass beyond the formal aspects of the theory. We will call them the *vicinity* and *smoothness assumptions* and show that they embody mathematical and physical requirements.

2. The Vicinity Assumption

The *vicinity assumption* incorporates at least two ideas. Physically, it begins with the idealization of the figure of a uniformly rotating Earth being roughly spherical. More precisely, one regards the Earth as being topologically equivalent to a sphere which includes the shape being that of a sphere, a spheroid, or an ellipsoid. One then asks under what circumstances can one take the surfaces of Σ to be *closed surfaces*? Intuitively, the answer is obvious: the equipotential surfaces must lie 'near' the surface of the Earth. However, a much more precise answer has long been known -- but almost forgotten!

The answer was given in 1901 by Paolo Pizzetti (1860-1918), who from 1900-1918 was Professor of Higher Geodesy at the University of Pisa. In PIZZETTI [4], he proved that when the maximum and mean radii R_{max}, R_m of the Earth are subject to the inequality

$$(1) \qquad R_{max} - R_m < \frac{1}{100} R_m ,$$

with ratio m of the centrifugal and gravitational accelerations being

$$(2) \qquad m < \frac{1}{280} ,$$

then the surfaces of Σ are *closed* for altitudes h above the surface of the Earth which satisfy the *Pizzetti inequality*:

$$(3) \qquad 0 < h < 5R_m .$$

He also observed that (3) remains valid when lunar tidal effects are involved, i.e. $m < \frac{1}{289}$. The coefficient of the upper bound in (3) to 4-place accuracy is 4.8930, but of course Pizzetti's values for the geophysical parameters are

now obsolete. However, upon taking latest accepted I.A.G. values given by Moritz, (1988), with $m < \frac{1}{290}$ to 3-place accuracy this coefficient is 4.969. The choice of the coefficient 5 in (3) is of no particular significance except it works! The essential idea is that on the basis of very general assumptions PIZZETTI was able to establish such an inequality.

In [5] W.D. Lambert presented an excellent qualitative discussion of a similar situation, and traced such considerations back to P.S. Laplace and his "Traité de Mécanique Céleste, Livre III" (see Chapter VII of the Bowditch translation of 1832). Laplace analyzed the case of the equipotential surface 'marking the outermost limit of the atmosphere' and determined its equatorial radius to be 6.6 times the radius of the Earth. Lambert did not make reference to the work of Pizzetti.

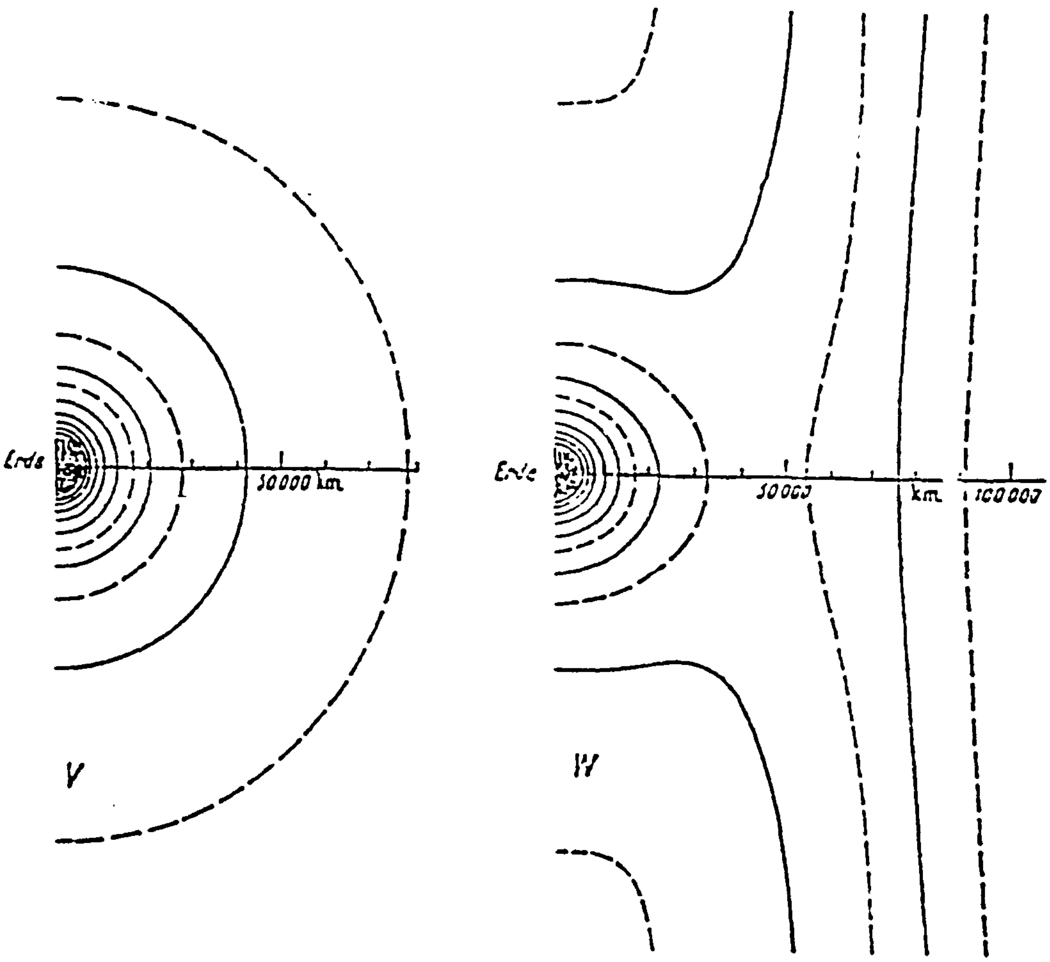

Figure 1. The gravitational potential V is illustrated in the left figure, the gravity potential W in the right figure. The corresponding equipotential surfaces for V and W are indicated by solid and dashed lines. This illustration is taken from page 537 of an article by K. JUNG in the Handbuch der Physik, Bd. 47 (Springer-Verlag, 1956).

It is immediate that when (3) holds the family Σ is not only closed, but also *bounded*. If we restrict our considerations to non-relativistic effects, and assume the space in the neighborhood of the Earth is 3-dimensional and Euclidean, then each $S \in \Sigma$ is a *compact* surface. We will return to this important topological property in a moment. Henceforth we will consider only such $S \in \Sigma$.

The second aspect of the vicinity assumption is to emphasize that almost all of the analysis involving differential geometric and tensorial techniques is local and valid only on a piece, viz in the vicinity of a point P, of a surface $S \in \Sigma$. Stated more precisely, these considerations hold only on an open connected subset, i.e., a *domain*, of S. When this is provided with a coordinate system x^r this domain becomes a *coordinate neighborhood*, or *chart* about the point P. Hence, in general, coordinates are local in character, i.e., valid only in a domain of S, and in order to obtain a "coordinatization" of even a piece of S it is necessary to employ several charts. The charts must be compatible, i.e. "mesh-together," and we will return to this in Section 3. These charts are said to *cover* the piece of S, and by virtue of the compactness of each $S \in \Sigma$ the Heine-Borel-Lebesgue covering property asserts that the entire surface S can be covered by a *finite number* of charts. This is illustrated by taking S to be 2-dimensional sphere S_2 where *two* charts are required to "patch up" the coordinate singularities which occur at the North and South poles. An even better covering of S_2, which avoids equatorial singularities, employs *six* charts. In Section 4 we will see that such singularities are *not* the result of an inappropriate choice of coordinates, but a topological property of S_2 and surfaces which are topologically equivalent to it.

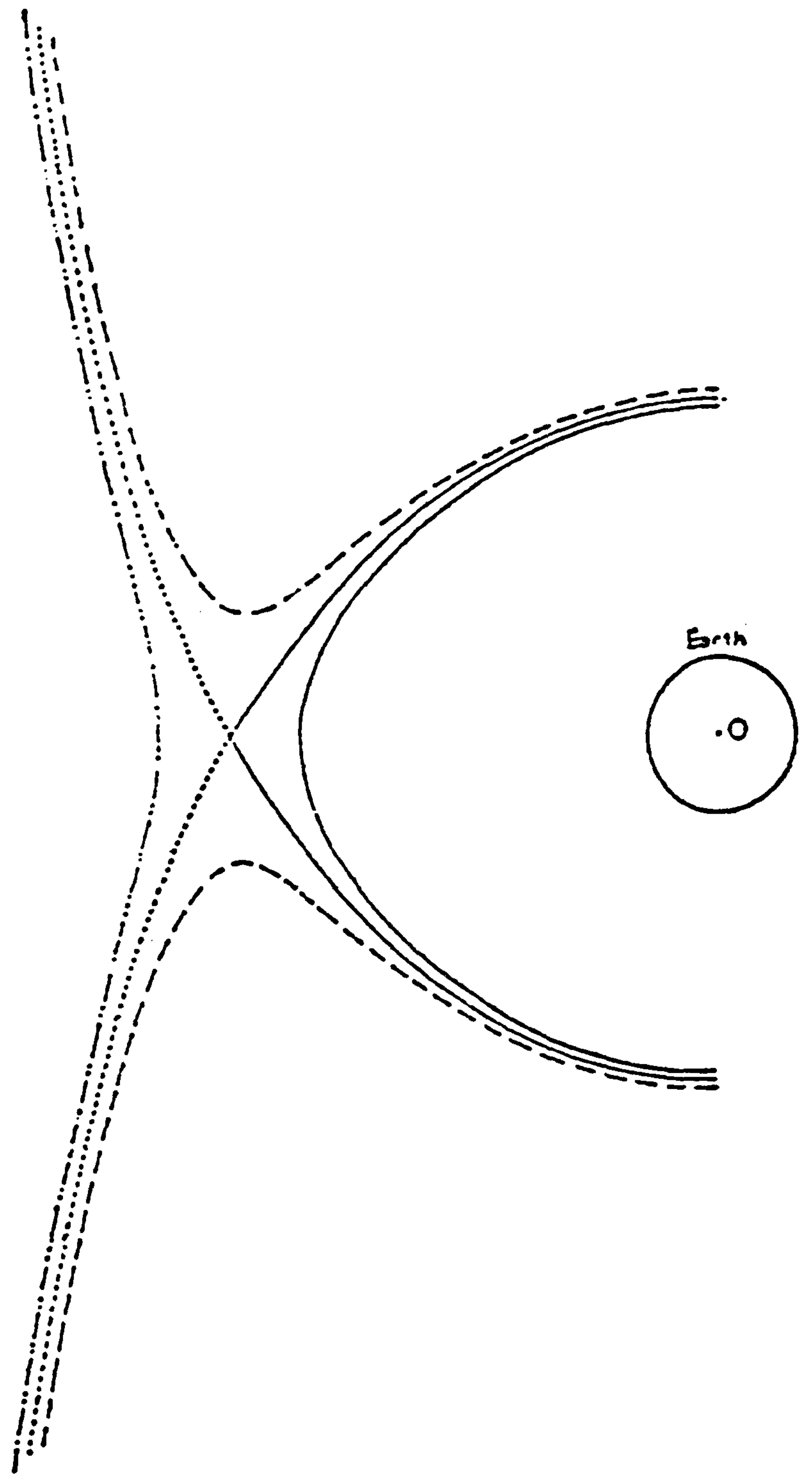

Figure 2. This figure is entitled "Limiting surfaces of the atmosphere and adjacent level surfaces ..." and is taken from LAMBERT [5]. Solid lines denote limiting equipotential surfaces, dashed lines denote analytical continuations (for a detailed description see [5]).

3. The Smoothness Assumption

The *smoothness assumption* refers to the degree of differentiability of the functions and coordinates occurring in our analysis. We briefly review the relevant notions for functions defined on a domain Ω of a 3-dimensional Euclidean space E_3.

A function F defined on Ω is said to be of *class* C^k for an integer $k \geq 0$ whenever F and all its partial derivatives up to and including those of order k are continuously differentiable with respect to the coordinates x^r on Ω. In this case we write $F \in C^k(\Omega)$, or simply $F \in C^k$ when Ω is evident. The class C^0 corresponds to merely continuous functions and C^ω indicates (real) analytic functions. In general $0 \leq k < \infty < \omega$ and if no specific value $k > 0$ is indicated, it is convenient to merely say that the functions are *smooth*. An important generalization of these ideas was introduced by E. Hölder (1882) in his classical work on potential theory, and we will also employ the Hölder class $C^{k,\ell}$ for $k \geq 0$ and $0 < \ell \leq 1$ in which the kth order partial derivatives satisfy a uniform Hölder condition of order ℓ. If $0 < \ell < 1$ membership in $C^{k,\ell}$ is weaker than differentiability, but stronger than mere continuity.

Quite obviously the first aspect of the smoothness assumption is to insure that all the functions belong to the appropriate classes in order that the various operations performed on them are meaningful. These requirements are not stated in [1], [2] or [3], and often they are likewise taken for granted in texts on differential geometry and tensors. Such a naive and uncritical approach is often harmless, but in the case of surface theory one clearly wants S to admit continuously varying tangent planes, normal vectors, principal curvatures, etc. and none of these properties is trivial or automatically true. These considerations will be discussed further in Section 4.

An even more important aspect of the smoothness assumption is to ascertain when the data obtained from geodetic measurements is sufficiently smooth to allow us to even meaningfully speak of a piece of S. The irregularities in the equipotential surfaces of Σ are dramatically shown in the beautiful computer-generated pictures recently given by P.J. MELVIN, [6]. While we may assume our current data, or our extrapolations/approximations of it for a $S \in \Sigma$, are adequate for an application of differential geometric techniques, as a first step we should unequivocally establish the precise nature of these requirements.

4. Local Surface Theory

The notion of a smooth $S \in \Sigma$ in E_3 is very complicated. Technically it is that of a 2-dimensional smooth oriented Riemannian manifold isometrically imbedded in E_3. The process of developing the requisite ideas is lengthy. It begins with a connected topological space X (consisting of suitable collections of sets called a *topology*) in which the elements of the sets are *points*. One next imposes axioms of separation and separability (the *Hausdorff* and *Second Countability Axioms*) in order to obtain abstract Euclideanlike properties. Coordinates are introduced by making X into a *topological manifold*, and the dimension of the manifold is obtained as a byproduct of this procedure. Locally, a topological manifold X_n looks like a domain of E_n. The apparatus of differential calculus is obtained only when X_n is provided with a *differentiable structure* and becomes a *differentiable manifold* M_n. Finally, M_n must be supplied with a smooth *Riemannian structure*, given an orientation, and the resulting *smooth Riemannian manifold* V_n imbedded in an Euclidean E_m.

In his report to the Third Hotine Symposium (Siena, 1975), GROSSMAN [7] gave a delightful discussion of the possibilities and difficulties. One of

his comments relative to the above procedure bears repeating since it is just as true now as it was then. He said

> "The predominant view is that space near the Earth is a manifold. The unpleasant fact of life is that no one has ever described a method for coordinatizing that supposed manifold in a way consonant with physical reality."

Stated more succinctly we can paraphrase this by saying that although the procedure, or predominant view, is clear it remains an open challenge to implement it in practice. Until this is done, much of differential geodesy is at best wishful thinking, and at worst a mathematical fiction.

Keeping Grossman's comment in mind, let us survey the situation in view of the vicinity and smoothness assumptions.

The first step is to introduce a Gaussian parametrization of the Euclidean coordinates x^r on a piece of S. By the Implicit Function Theorem this amounts to solving the implicit equation of $S : F(x^1, x^2, x^3) = 0$ for *one* of the coordinates and introducing an arbitrary set of parameters $u^\alpha = (u^1, u^2)$ into the coordinate chart on S. For those S which are topologically equivalent to a 2-sphere S_2 this entails a square root involving the other two coordinates, and choosing the *two* signs on this radical yields *two* coordinate charts on S. The above process can be done in *three* different ways (one for each x^r) and this leads to the *six* charts mentioned previously.

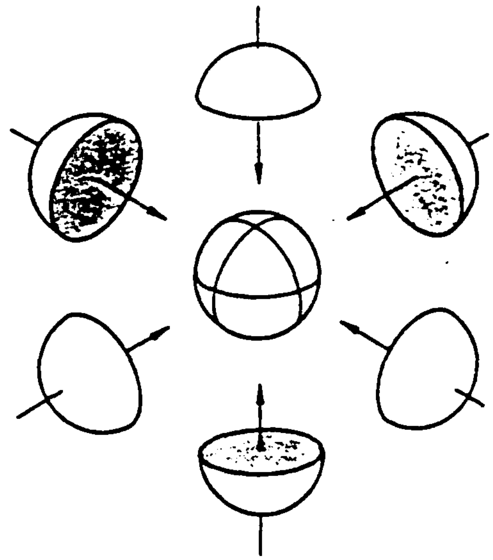

Figure 3. The covering of the 2-sphere S_2 by six charts as given in the textbook Differential Geometry of Curves and Surfaces by M. DO CARMO, page 56 (Prentice-Hall, 1976). For more details, see our Appendix, equations (A.1)-(A.3).

Suppose this has been done, and that a piece of S is locally given a Gaussian parametrization

$$x^r = x^r(u^1, u^2) \ . \tag{4}$$

Then the u^α are regarded as independent parameters on the coordinate chart (U, x^r) of S, and this is said to be a C^k $(k > 0)$ piece of S whenever the coordinates x^r in (4) are C^k functions of the parameters u^α. This is written, $x^r \in C^k$ and S is locally a C^k*-surface* whenever the coordinate charts covering S are selected in a manner such that a common value of k can be assigned to *all* the charts covering S. As mentioned in Section 2, by the compactness of each $S \in \Sigma$, a finite number of charts suffices to cover each $S \in \Sigma$. Let u^α and $\bar{u}^\alpha$ denote the parameters in a pair of charts

whose intersection $U \cap \bar{U}$ contains an arbitrary point $P \in S$. Then the u^α and $\bar{u}^\alpha$ are related by the parameter changes

$$(5) \qquad \begin{aligned} \bar{u}^\alpha &= f^\alpha(u^1, u^2) , \\ u^\alpha &= g^\alpha(\bar{u}^1, \bar{u}^2) , \end{aligned}$$

where f^α, $g^\alpha \in C^k$. In other words, the local parameter descriptions are reversible and C^k-related to each other. In order to insure that outward pointing unit normal vectors $\underset{\sim}{\nu}$ on S retain their outward pointing character, it is necessary that S be *orientable*. This requires that the functional determinants have a fixed sign, viz an *orientation*, say

$$(6) \qquad \frac{\partial(\bar{u}^1,\bar{u}^2)}{\partial(u^1,u^2)} > 0 ,$$

and this guarantees reversibility of the parametrizations in (5).

The C^k-character of (5) is inherited by the coordinates x^r in (4). The Inverse Function Theorem then requires that *at least one* of the functional determinants

$$(7) \qquad \frac{\partial(x^1,x^2)}{\partial(u^1,u^2)} , \quad \frac{\partial(x^2,x^3)}{\partial(u^1,u^2)} , \quad \frac{\partial(x^3,x^1)}{\partial(u^1,u^2)}$$

be non-vanishing, and this is tantamount to the linear independence of the tangent vectors $\partial x^r/\partial u^1$, $\partial x^r/\partial u^2$ on the piece of S, i.e., to the non-vanishing of a component ν^r of $\underset{\sim}{\nu}$. If *all three* of the determinants in (7) vanish then the Inverse Function Theorem fails to hold, and the x^r are said to have a *singularity*. Some of these may be artificial, i.e., due to a poor choice of parameters, while others are of a topological nature and cannot be avoided. This is responsible for the non-existence of a single global singularity-free coordinate system on S_2, and explains the necessity of several local coordinate charts on S_2 and its topological equivalents. In effect, ultimately this is a byproduct of a celebrated theorem of L.E.J.

Brouwer (1909-10), see J. MILNOR, [8] and [9], that an n-sphere S_n admits a globally defined field of non-vanishing smooth tangent vectors if and only if n is *odd*. We will return to this question at the end of Section 5.

The above discussion has indicated the meaning and delicacy attached to the requirement that S be a C^k-surface. In the remainder of this section it is understood that references to S are strictly local in character, and refer only to a piece of S. We now determine the smoothness requirements on the various geometric quantities associated to S in Gaussian differential geometry.

Mathematically, the most striking property of S is that it is *imbedded* in E_3. Let x^r be Cartesian coordinates on a domain $\Omega \subset E_3$, with δ_{rs} being the components of the constant *Euclidean metric tensor*, i.e., the "infinitesimal" square of the distance between a pair of points $P, Q \in \Omega$ is given by

$$ds^2 = \delta_{rs} dx^r dx^s \ . \tag{8}$$

If now u^α are local parameters on a surface S in E_3 with $P, Q \in S$, then we also have (4) and for $P, Q \in \Omega \cap S$:

$$ds^2 = a_{\alpha\beta} du^\alpha du^\beta \tag{9}$$

where $a_{\alpha\beta}$ are the components of the *surface metric tensor*, i.e. *the coefficients of the first fundamental form*. The agreement of (8) and (9) on $\Omega \cap S$ immediately yields

$$a_{\alpha\beta} = \delta_{rs} \frac{\partial x^r}{\partial u^\alpha} \frac{\partial x^s}{\partial u^\beta} \tag{10}$$

which is not only the defining equation for the $a_{\alpha\beta}$, but also the *basic imbedding equation*. The equality of these two expressions for ds^2 is the identity of the notions of distance on E_3 and S. This is what is meant by an *isometric imbedding* of S in E_3. It is a nontrivial requirement since it

involves S being a subspace of E_3 and possessing both a metric and topology induced by those of E_3. For the moment, let us suppose that this imbedding has been achieved -- we will consider it in more detail at the end of this section.

Since S is a C^k-surface $(k \geq 1)$ by definition the $x^r \in C^k$, and hence by virtue of (10) it immediately follows that the $a_{\alpha\beta} \in C^{k-1}$. Moreover, since the determinant $a \equiv \det\|a_{\alpha\beta}\|$ is a polynomial function it follows that a, $\sqrt{a}$, and the *Levi-Civita surface dualizors* $\epsilon_{\alpha\beta}$, $\epsilon^{\alpha\beta}$ are of class C^{k-1}. By inspection of their definitions, the components ν^r of the unit normal $\underset{\sim}{\nu}$ to S are also of class C^{k-1}, and $\Gamma^{\alpha}_{\beta\gamma} \in C^{k-2}$. The *coefficients* $b_{\alpha\beta}$ *of the second fundamental form*, the determinant $b \equiv \det\|b_{\alpha\beta}\|$, as well as the *Gaussian (total) curvature* K, the *Germain (mean) curvature* H, and the higher order fundamental tensors (see [10]) are all of class C^{k-2}. Hence, when $k \geq 3$ all the basic quantities occurring in surface theory are smooth!

Such a smoothness requirement allows one to answer the question of the local existence and uniqueness of a surface S which is *assumed* to be isometrically imbedded in E_3. O. Bonnet (1867) proved the following:

Fundamental Theorem of Local Surface Theory

Let the $a_{\alpha\beta}$, $b_{\alpha\beta}$ be defined on a chart of $S \subset E_3$. Then a necessary and sufficient condition that a pair of quadratic differential forms having these tensors as coefficients, be the first and second fundamental forms of S is that locally S be of class C^3, with

(i) $$a > 0 ,$$

and that these tensors satisfy

(ii) the *Gauss equation*

$$R_{\alpha\beta\lambda\mu} = b_{\alpha\lambda} b_{\beta\mu} - b_{\alpha\mu} b_{\beta\lambda} ,$$

and

(iii) the *Codazzi-Mainardi equations*

$$b_{\alpha\beta,\gamma} - b_{\alpha\gamma,\beta} = 0 .$$

Moreover, if (i)-(iii) are satisfied, then locally a C^3-surface S is uniquely determined up to position, i.e., a rotation and translation, in E_3.

Both (ii) and (iii) are integrability conditions of a system of differential equations known as the *Gauss formulae*, and (ii) is essentially the *Theorema Egregium* of Gauss. An obvious question is whether Bonnet's theorem can be improved, i.e., can the differentiability conditions be weakened? In other words, can these results be established for C^2-surfaces? This would be attractive since in such a situation: $x^r \in C^2$, $a_{\alpha\beta}$ and $\nu^r \in C^1$, and $b_{\alpha\beta} \in C^0$, which would still allow for a continuously differentiable tangent plane and normal $\underset{\sim}{\nu}$. This was done by P. Hartman and A. Wintner (1950) who by an ingenious argument replaced (ii) and (iii) by integral equations. No other improvement of the theorem is known, or seems possible.

We now return to the question of the existence of a local isometric imbedding of S in E_3. By definition this involves finding a *singularity-free solution* $x^r(u^1, u^2)$ of (10), which is said to provide a *realization* of $a_{\alpha\beta}$. This requires regarding (10) not as a definition for $a_{\alpha\beta}$, but as a system of three non-linear first order partial differential equations for the x^r when the $a_{\alpha\beta}$ are *specified*. The obvious procedure is to appeal to the well-known Cauchy-Kovalevska Theorem; however this requires S to be a (real) analytic surface, i.e., x^r and $a_{\alpha\beta} \in C^\omega$. This is a severe requirement, but unfortunately a common one in pure mathematics!

Suppose for the moment we generalize the situation and replace S by an

n-dimensional Riemannian manifold V_n, and E_3 by E_m. Then the Greek indices range over the values 1 to n, and the Latin indices over 1 to m in (10). L. Schläfli (1873) conjectured that in this case, since $a_{\alpha\beta}$ has $n(n + 1)/2$ independent components, it suffices to take $m = n(n + 1)/2$. This was proven by M. Janet (1926), and some missing details were later supplied by C. Burstin (1931). Hence we have

<u>The Fundamental Theorem on Local Isometric Imbedding</u>

Any Riemannian manifold V_n having a C^{ω}-metric tensor can be locally isometrically imbedded analytically in an E_m with $m = n(n + 1)/2$.

Hence, when $V_2 = S$, so that $n = 2$, the problem is solved for C^{ω}-surfaces by taking $m = 3$. It has proved remarkably difficult to replace the stringent requirement of analyticity by weaker conditions of differentiability. The best result in this direction is due to H. JACOBOWITZ [11], who proved the following:

<u>Weakened Fundamental Theorem on Local Isometric Imbeddings</u>

Let S be a surface which admits a metric $a_{\alpha\beta} \in C^{k,\ell}$ with $k \geq 3$ and $0 < \ell < 1$, and which possesses a local $C^{k,\ell}$ realization in E_m. Then

(i) if S has Gaussian curvature $K > 0$, then $m = 3$, and

(ii) with no restrictions on the sign of K, then $m = 4$.

It is likely that this is the best possible result, since A.V. Pogorelov (1971) constructed a S having a $C^{2,1}$ metric $a_{\alpha\beta}$ which admits no C^2-realization in E_3. The classic example is that of D. Hilbert (1901), viz. a C^{ω}-surface having constant $K = -1$, which cannot be analytically realized in E_3 without the occurrence of singularities.

The above discussion is intended to illustrate that the question of a local isometric imbedding of S in E_3 is by no means either obvious or even

possible.

5. Topological Considerations

In Section 4 we have indicated the steps involved in the notion of a smooth surface $S \in \Sigma$ which is imbedded in an E_3. We now discuss how various global topological notions enter into the picture.

E_3 is a topological space, and its usual topology τ is Hausdorff and possesses a countable basis. We denote this topological space by (E_3, τ). Since S is topologically a *subspace* of (E_3, τ), its induced topology is *also* Hausdorff and second countable. By Pizzetti's result, each S is closed and bounded, hence since Σ is in E_3, each $S \in \Sigma$ is *compact*.

E_3 is also a 3-dimensional topological *manifold*, and each S is a 2-dimensional topological *submanifold* of E_3. Moreover, both E_3 and each S are *metric spaces*, which we denote by (E_3, ρ) and (S, σ), where ρ and σ are respectively the *metrics*, i.e. *distance functions*, on E_3 and S. The metrics ρ and σ then generate the usual topologies on E_3 and S, with σ being induced by ρ. It can be shown that E_3 and each S can be provided with compatible differentiable structures and hence E_3 and S are *differentiable manifolds*. When provided with the *Riemannian metrics* δ_{rs} and $a_{\alpha\beta}$ (the former being regarded as a *flat* Riemannian metric), both E_3 and each S are *Riemannian manifolds*. As a metric space, the *distance* $\sigma(P, Q)$ between a pair of points $P, Q \in S$ is given by the *infimum* of lengths $L(C)$ of all piecewise smooth curves joining P and Q.

Our immediate goal is to show that each (S, σ) is a *complete* metric space. By the property of completeness we mean that every Cauchy sequence of points on S is a convergent sequence. Stated intuitively, completeness guarantees that every sequence of points on S which tries to converge succeeds in the sense that it finds a point in (S, σ) to converge to. In a

moment we will translate this into a geometric/geodetic property of S.

The completeness of (S, σ) is easily established. First, a standard theorem in topology states that a compact metric space is automatically a complete metric space. An alternate argument is that the metric space (E_3, ρ) is complete, and that a subspace (S, σ) of (E_3, ρ) is complete whenever S is closed in E_3. But this is guaranteed by Pizzetti's result! Hence the metric space (S, σ) is complete.

We now assume that each (S, σ) is a *connected* space. This means that each S consists of a single piece, i.e. it is not the union of a pair of disjoint pieces. Such a requirement is obvious physically, for if it were not true there would be no possibility of moving from point to point on S. The geometric/geodetic consequences of completeness are then given in the following result of H. HOPF and W. RINOW [12].

Hopf-Rinow Theorem

If a connected surface S is complete as a metric space, then any two points P and Q of S can be joined by a unique geodesic Γ which has length $\sigma(P, Q)$ and is the curve of shortest length joining these points.

Indeed a classical result of O. Bonnet (1855) goes even further, and provides a bound on $\sigma(P, Q)$:

Bonnet's Theorem

On a compact surface S having Gaussian curvature K, $0 < k_0 \leq K$, where k_0 is a constant, the distance is bounded by

$$\sigma(P, Q) \leq \pi/\sqrt{k_0} .$$

In [1] Marussi stated that a goal of his intrinsic geodesy was to carry out geodetic investigations *without* imposing any additional hypotheses relative to the selection of a particular spheroid or ellipsoid. The Hopf-Rinow theorem, which essentially involves translating a geometric/geodetic

situation into a topological setting, assures us that geodesics on the equipotential surfaces $S \in \Sigma$ are indeed curves of minimal length!

We are now in a position to obtain a rather sharp global topological characterization of all $S \in \Sigma$. Assume that each S has $K > 0$ (with K not necessarily constant) or, equivalently, that each S is a convex surface. Geodetically, such surfaces are characterized by the fact that on them plumb lines (i.e. the outward pointing normals) at neighboring points do not intersect. By our previous analysis we know that each S is a complete surface. In E_3 we then have the following classification theorem of A.D. ALEKSANDROV [13].

Aleksandrov Theorem

> Every complete convex surface S in E_3 is topologically equivalent to either a 2-sphere S_2, a plane, or a cylinder.

The latter two possibilities have $K = 0$, and hence can be eliminated. Thus all our $S \in \Sigma$ are topologically equivalent, i.e. *homeomorphic*, to a 2-sphere S_2. This equivalence includes spheres, spheroids, ellipsoids, as well as "bumpy" spheroids, and each of these satisfies the topological properties of being compact and connected as well as being complete and orientable. Obviously S_2 is trivially a topological space, a metric space, a topological manifold, and is a subspace of E_3. Moreover, S_2 can be provided with a *unique* differentiable structure, a smooth Riemannian metric, and as a Riemannian manifold it is isometrically imbedded in E_3. Hence each $S \in \Sigma$ is not only homeomorphic but also differentiably equivalent, i.e., *diffeomorphic*, to S_2. In other words, topological transformations of S are not only continuous but smoothly differentiable. Note that although diffeomorphisms are homeomorphisms, i.e., differentiability implies continuity, the converse is not true.

We may summarize these considerations in the following.

Theorem

All the closed equipotential surfaces Σ of an idealized sphere-shaped uniformly rotating Earth, i.e., those which satisfy the Pizzetti inequality (3), are diffeomorphic to a 2-sphere S_2.

Our argument has been based on translating the following *five* physical requirements:

(i) the space near the Earth is Euclidean and 3-dimensional;

(ii) each $S \in \Sigma$ consists of one piece;

(iii) one can measure distances between different points on each $S \in \Sigma$;

(iv) each $S \in \Sigma$ admits an upward pointing direction;

(v) neighboring plumb lines on each $S \in \Sigma$ do not intersect;

into precise topological and geometric requirements.

Finally, the Gauss-Bonnet theorem for S yields

$$\oint\!\!\oint_S K dA = 2\pi\chi(S) \quad ,$$

where dA is the *area element* on S and $\chi(S)$ is the *Euler-Poincaré characteristic* on S. For S topologically equivalent to S_2, $\chi(S) = 2$. The significance of this quantity is given by the

Poincaré-Hopf Theorem

If $\underset{\sim}{v}$ is a smooth vector field on a compact orientable surface S, and $\underset{\sim}{v}$ has only a finite number of zeros, then the sum of the indices of singularity j of $\underset{\sim}{v}$ is equal to $\chi(S)$.

The result $\chi(S) = \Sigma j = 2$ then explains the inevitability of the singularities of a smooth tangent vector on S, as mentioned in Section 3. Typically one has *two* singularities on S, viz., a pair of sources, sinks, or centers with *each* singular point having a singularity index $j = +1$. The diversity of possible singularities involving saddle points, dipoles, etc.,

but with the sum of the indices being constrained by $\chi(S) = 2$, also indicates the various choices of coordinate systems having 'simpler' or more 'severe' types of singularities.

The intent of our discussion has been to show that the mathematical foundations of the Marussi-Hotine approach to geodesy are both rich and intricate. A failure to succeed in resolving a particular question may not be attributable merely to a lack of ingenuity or initiative -- a solution need not *exist*! Lord Butler once said that "politics is the art of the possible." We would suggest that a similar remark is apropos in differential geodesy, where *topology* is the art of the possible.

Acknowledgement.

It is a pleasure for me to express my gratitude to Bernard Chovitz for his critical comments and discussion on a preliminary draft of this manuscript. In particular, he provided me with a copy of Lambert's valuable paper [5].

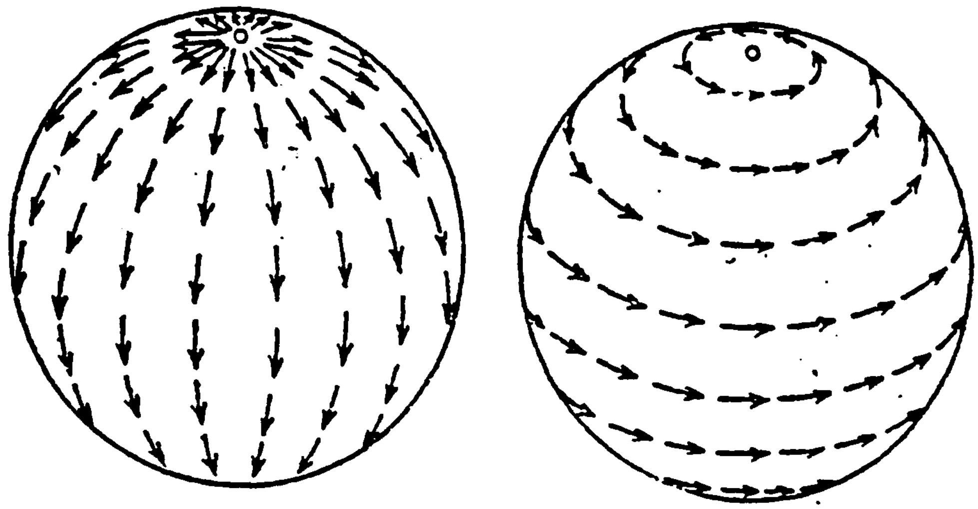

Figure 4. The singularities of the tangent vectors on the 2-sphere. In the left figure the latitudinal tangents have singular points at the North and South poles, each having an index $j = +1$ corresponding to a "source" and "sink" respectively (using the language of hydrodynamics). In the right figure the longitudinal tangents are illustrated. Notice that as the geographical parallels are moved northward or southward, one obtains singularities again having index $j = +1$ corresponding to "vertices" at the North and South poles. The values of the indices are not direction dependent! This figure is taken from Intuitive Topology (in Russian) by V.G. BOLTYANSKII and V.A. EFREMOVICH, page 68, (Nauka, 1982).

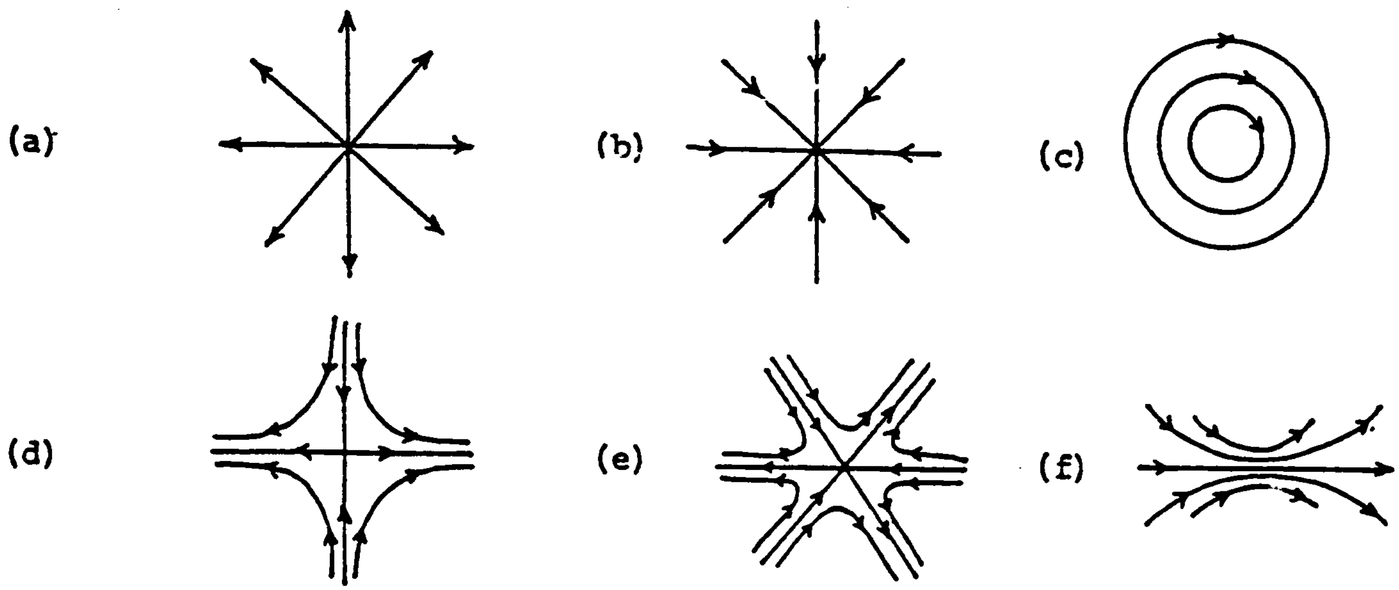

Figure 5. Examples of singular points of vector fields, and their indices. The indices of these singularities are $j = +1$ in (a), (b), (c); $j = -1$ in (d), $j = -2$ in (e) and $j = +2$ in (f). This figure is taken from H. HOPF: Differential Geometry in the Large, Lecture Notes in Mathematics 1000, page 12 (Springer-Verlag, 1983).

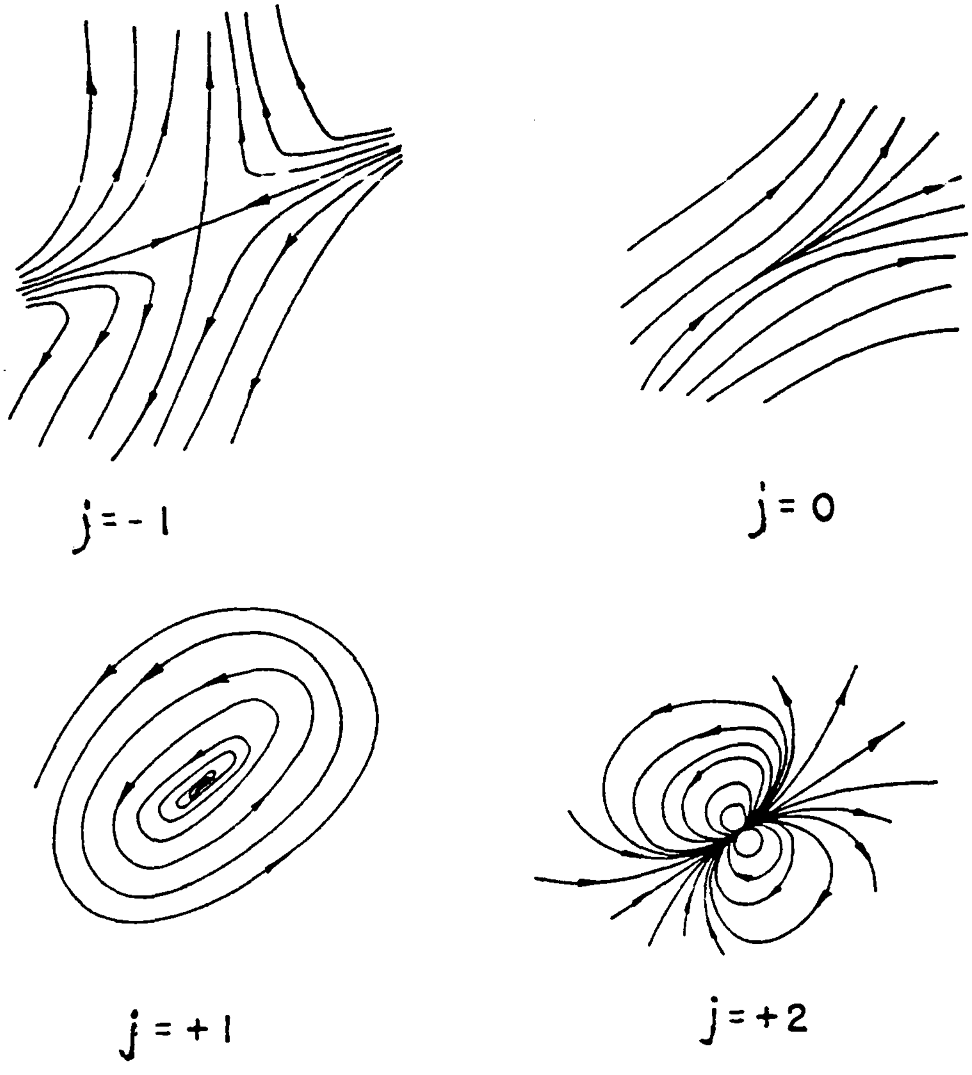

Figure 6. Examples of singular points of vector fields, and their indices. This shows that the curves, i.e. the trajectories of the vector fields, need not be as regular-shaped as illustrated in Figure 5. This figure appears in Topology from the Differential Viewpoint by J.W. MILNOR, page 33 (The University Press of Virginia, 1965).

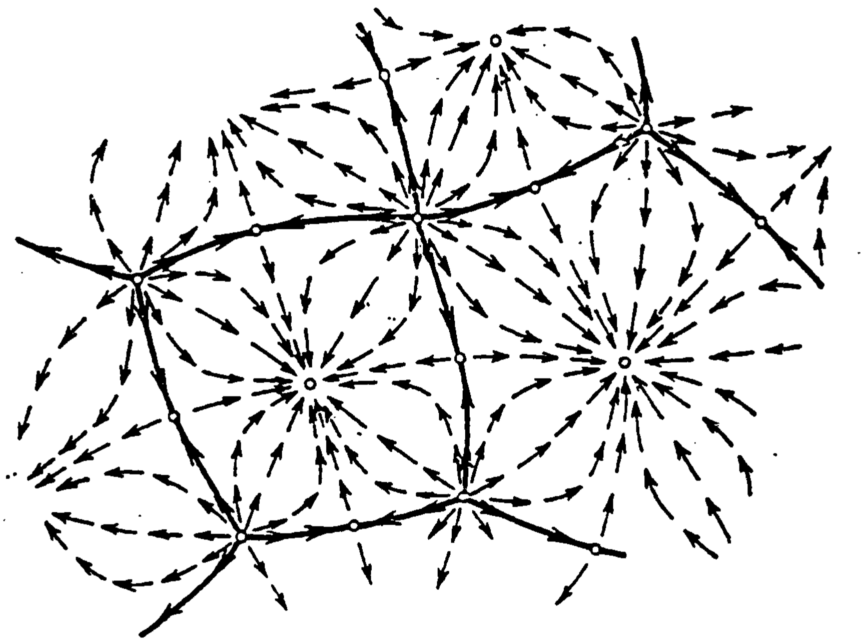

Figure 7. An example of a number of singular points occurring in a single chart, say on a surface of irregular shape. This figure occurs on page 73 of the Russian textbook cited in the caption of Figure 4.

Appendix

In this Appendix we will illustrate some of the ideas in Sections 4 and 5 for the case of a 2-sphere S_2. None of the results are new, but the approach may be useful in showing how the calculation are done when topological notions are taken into account.

For the 2-sphere S_2 the implicit equation is given by

$$F(x^1, x^2, x^3) \equiv (x^1)^2 + (x^2)^2 + (x^3)^2 - 1 = 0 \tag{A.1}$$

where

$$\underset{\sim}{x} = x^r = (x^1, x^2, x^3)$$

denote Cartesian coordinates in Euclidean 3-space E_3. Let

$$\Delta_{jk} \equiv \sqrt{1-(x^j)^2-(x^k)^2} \tag{A.2}$$

denote the solution of (A.1) for *one* of the coordinates, say x^i, in terms of the other *two*, viz. x^j and x^k with i, j and k being different. Then upon taking into account the choice of signs on the radical in (A.2), and the fact one can solve for the Δ_{jk} in *three* different ways, we obtain the following *six* charts on S_2:

$$\begin{aligned}
(U_1, \underset{\sim}{x}) &= (x^1, x^2, +\Delta_{12}) \\
(U_2, \underset{\sim}{x}) &= (x^1, x^2, -\Delta_{12}) \\
(U_3, \underset{\sim}{x}) &= (x^1, +\Delta_{13}, x^3) \\
(U_r, \underset{\sim}{x}) &= (x^1, -\Delta_{13}, x^3) \\
(U_5, \underset{\sim}{x}) &= (+\Delta_{23}, x^2, x^3) \\
(U_6, \underset{\sim}{x}) &= (-\Delta_{23}, x^2, x^3) .
\end{aligned} \tag{A.3}$$

The inclusion of $\underset{\sim}{x}$ in the above equations is necessary to indicate the

coordinatization used on the neighborhoods. Note that the union of the coordinate neighborhoods U_1 and U_2, denoted by $U_1 \cup U_2$ covers S_2 except for the equator

$$(x^1)^2 + (x^2)^2 = 1,$$

and it is easy to see that by forming the sixfold union $U_1 \cup U_2 \cup \cdots \cup U_6$ we obtain a covering of all of S_2. The six coordinate neighborhoods are illustrated in Figure 3.

Using Hotine's notation the Gaussian parametrization on the 2-sphere S_2 is given by

$$u^\alpha = (\omega, \phi) \tag{A.4}$$

where ω is the longitude, and ϕ is the latitude as illustrated in Figure 4. We take these to range over the open intervals:

$$0 < \omega < 2\pi, \quad -\pi/2 < \phi < \pi/2 \tag{A.5}$$

and hence we have *excluded* the meridian $\omega = 0$ from our parametrization. Note that this meridian also contains the North pole

$$N : (\omega = 0, \ \phi = \pi/2)$$

and the South pole

$$S : (\omega = 0, \ \phi = -\pi/2).$$

If we changed the parametrization (A.4) to

$$\bar{u}^\alpha = (\bar{\omega}, \bar{\phi}) \tag{A.6}$$

where $\bar{\omega}$ remains the longitude, and $\bar{\phi}$ is the colatitude, i.e.

$$\begin{aligned} \bar{\omega} &= \omega \\ \bar{\phi} &= \pi/2 - \phi \end{aligned} \tag{A.7}$$

with

$$0 < \bar{\omega} < 2\pi, \quad 0 < \bar{\phi} < \pi , \tag{A.8}$$

then

$$\frac{\partial(\bar{\omega},\bar{\phi})}{\partial(\omega,\phi)} = -1 . \tag{A.9}$$

This sign is opposite that employed in (6), however all that was required is that the functional determinant have a fixed sign.

In terms of the Cartesian coordinates $\underset{\sim}{x} = x^r = (x^1, x^2, x^3)$ and the parametrization (A.4) and (A.5) we have the following parametrization:

$$\text{(A.10)} \qquad \begin{aligned} x^1 &= \cos\omega \cos\phi , \\ x^2 &= \sin\omega \cos\phi , \\ x^3 &= \sin\phi , \end{aligned}$$

which by (A.5) *omits* the North and South poles N and S respectively, i.e. (0, 0, 1) for $\phi = \pi/2$ and (0, 0, -1) for $\phi = -\pi/2$.

By direct calculation we have

$$\text{(A.11)} \qquad \begin{aligned} \frac{\partial(x^1,x^2)}{\partial(\omega,\phi)} &= \cos\phi \sin\phi , \\ \frac{\partial(x^2,x^3)}{\partial(\omega,\phi)} &= \cos\omega \cos^2\phi , \\ \frac{\partial(x^3,x^1)}{\partial(\omega,\phi)} &= \sin\omega \cos^2\phi ; \end{aligned}$$

and the requirements of the Inverse Function Theorem are satisfied. Note that at the excluded poles N and S, all *three* of these functional determinants are identically zero.

Thus, the *partial* covering of S_2 by coordinate neighborhoods $(U, \underset{\sim}{x})$ consists of a pair of right and left hemispheres which *omits* the meridian of reference $\omega = 0$.

Corresponding to the parametrization (A.10) the tangent vectors to the longitude and latitude are respectively given by

$$\partial x^r/\partial\omega = (-\sin\omega\cos\phi,\ \cos\omega\cos\phi,\ 0)$$

(A.12)

$$\partial x^r/\partial\phi = (-\cos\omega\sin\phi,\ -\sin\omega\sin\phi,\ \cos\phi).$$

We denote these by $\underset{\sim}{x}_\omega$, $\underset{\sim}{x}_\phi$ respectively, and note that on the equator $\phi = 0$ we have

$$\underset{\sim}{x}_\omega = (-\sin\omega,\ \cos\omega,\ 0),$$

$$\underset{\sim}{x}_\phi = (0,\ 0,\ 1);$$

however at the excluded poles N and S

$$\underset{\sim}{x}_\omega = (0,\ 0,\ 0),$$

$$\underset{\sim}{x}_\phi = (\mp\cos\omega,\ \mp\sin\omega,\ 0).$$

This suggests that -- at least in this parametrization -- both N and S are singular points since the longitudinal tangent vector $\underset{\sim}{x}_\omega$ vanishes at these points.

By using (10) we obtain the following expression for the first fundamental form (9)

(A.13) $$ds^2 = \cos^2\phi\, d\omega^2 + d\phi^2,$$

i.e.

(A.14) $$\| a_{\alpha\beta} \| = \begin{Vmatrix} \cos^2\phi & 0 \\ 0 & 1 \end{Vmatrix}$$

with

(A.15) $$a = \cos^2\phi\ .$$

Note that $a > 0$, and $a = 0$ only at the exclude poles N and S.

The covariant components of the unit normal $\underset{\sim}{\nu}$ are defined by

(A.16) $$\nu_r = \frac{1}{2}\epsilon_{rmn}\, x^m_\alpha\, x^n_\beta\, \epsilon^{\alpha\beta}$$

and hence

(A.17) $$\underset{\sim}{\nu} = (\cos\omega\cos\phi,\ \sin\omega\cos\phi,\ \sin\phi).$$

On the equator $(\omega, \phi = 0)$ we have

$$\underset{\sim}{\nu} = (\cos \omega, \sin \omega, 0)$$

and curiously at the excluded poles N and S

$$\underset{\sim}{\nu} = (0, 0, \pm 1).$$

This is surprising since as noted previously the longitudinal tangent vanishes at both N and S. The result occurs since in (A.16) the Levi-Civita dualizor $\epsilon^{\alpha\beta}$ involves a factor of $1/\sqrt{a}$ which cancels with a common factor of $\cos \phi$ in the vector product of $\underset{\sim}{x}_{\omega}$ and $\underset{\sim}{x}_{\phi}$.

If we regard (A.10) as being the imbedding equations of S_2 in E_3, then the range (A.5) of values of ω, ϕ omits not only the reference meridian $\omega = 0$, but the poles N and S. Hence (A.10) cannot be taken as defining all of S_2 in E_3. The expressions (A.11)-(A.17) show no analytical difficulties along the reference meridian except at the excluded poles $\phi = \pm \pi/2$.

Likewise, computing the coefficients $b_{\alpha\beta}$ of the second fundamental form by

$$b_{\alpha\beta} = -\delta_{rs} \, x^r_\alpha \, \nu^s_\beta \tag{A.18}$$

we have

$$\| b_{\alpha\beta} \| = \begin{Vmatrix} -\cos^2\phi & 0 \\ 0 & 1 \end{Vmatrix} \tag{A.19}$$

and

$$b = \cos^2\phi . \tag{A.20}$$

Note that again $b > 0$, and $b = 0$ only at the excluded poles N and S. By taking the usual definitions of the Gaussian (total) and Germain (mean) curvature we find that

$$K = 1, \quad H = -1.$$

By direct calculation using the Gauss-Bonnet formula, since the area element of dA on S_2 is

$$dA = \cos\phi \; d\omega \; d\phi ,$$

we have

$$\oint\!\!\oint_{S_2} K \, dA = \int_{-\pi/2}^{\pi/2}\int_{0}^{2\pi} \cos\phi \; d\omega \; d\phi = 4\pi \; .$$

Hence, $\chi(S_2) = 2$ as claimed. This is a correct result, however the calculation should be done more carefully. We consider a partial covering of S_2 consisting of the Northern hemisphere

$$U_N : (x^1)^2 + (x^2)^2 + (x^3)^2 = 1, \; (x^3 \geq 0);$$

the Southern hemisphere

$$U_S : (x^1)^2 + (x^2)^2 + (x^3)^2 = 1, \; (x^3 \leq 0);$$

and denote the equator by

$$C_O : (x^1)^2 + (x^2)^2 = 1.$$

Then one calculates

$$\text{(A.21)} \qquad \begin{aligned} &\iint_{U_N} K \, dA + \oint_{C_O} \sigma \, ds \; , \\ &\iint_{U_S} K \, dA + \oint_{C_O} \sigma \, ds \end{aligned}$$

where σ is the geodesic curvature of C_O, ds is the element of arc length on C_O, and the line integral is taken in a sense such that each of the hemispheres remains on the left-hand side of C_O. In terms of the Gaussian parametrization we have

$$\text{(A.22)} \qquad U_N : \begin{array}{c} 0 \leq \omega \leq 2\pi \\ 0 \leq \phi \leq \pi/2 \end{array} \; ; \quad U_S : \begin{array}{c} -2\pi \leq \omega \leq 0 \\ -\pi/2 \leq \phi \leq 0 \end{array} .$$

Then each of the surface integrals is equal to 2π, and when the two expressions in (A.21) are added, the line integrals cancel. Actually since C_O is a great circle, then $\sigma = 0$ and the integrands of each of the line integrals is identically zero. This can be verified by using the classical

expression for σ:

$$\text{(A.23)} \qquad \sigma = \frac{1}{\sqrt{a_{11}a_{22}}} \frac{\partial}{\partial u^1} (\sqrt{a_{22}}) = \frac{1}{\cos\phi} \frac{\partial}{\partial \omega} (1) .$$

More generally for a parallel C_{ϕ_0}, $\phi = \phi_0$, of S_2 we have the geodesic curvature

$$\text{(A.24)} \qquad \sigma = \tan \phi_0$$

and applying this to the equator C_0 we get $\sigma = 0$. Thus, we obtain

$$\text{(A.25)} \qquad \oint\!\!\oint_{S_2} K\, dA = \iint_{U_N} K\, dA + \iint_{U_S} K\, dA = 4\pi .$$

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